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LETTER TO THE EDITOR

Squeezing and minimum uncertainty states in the supersymmetric harmonic oscillator

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Abstract. A displacement operator for a subset of the supercoherent states is found. It is also shown that a diagonal version of the squeezing operator reduces the fluctuations of one of the quadratures, but the reduction is not enough to go below the standard quantum noise level. New states are found, which are minimum uncertainty for the new operators X and P, and lead to coherence and squeezing, in the sense of equal fluctuations in the two quadratures and reduction in one of them, respectively. Moreover, these states are associated with the model of Jaynes and Cummings in the strong-coupling limit

Coherent states have been widely used in many areas of physics such as quantum optics and nuclear physics [1]. The main purpose of this letter is to construct squeezed states for supersymmetric systems. Although the calculations that will be presented here may appear academic, it is important to note that in recent years a vast amount of literature has been devoted to squeezing, the reason being an increasing interest in the optical detection of very small changes of a given physical quantity, which can be converted into a phase or frequency shift. The quantity to be measured is so small that the signal is below the quantum noise limit and squeezing is one possible answer. The typical examples we are bearing in mind are gravitational wave detection and the laser gyroscope. Also, in a series of recent experiments, squeezed states have been generated and observed in a non-linear optical system [2]. In the first part of this letter, we take Aragone and Zypman's supercoherent states [3] for the supersymmetric harmonic oscillator and show that the fermionic sector can be generated with a displacement operator, starting from vacuum. This sector can also be squeezed through the usual squeezing operator. For the sector orthogonal to the fermionic one, we show that the fluctuations of $\bar{x}$ and $\bar{p}$ can be reduced via a squeezing operator, but never below the quantum limit. The rest of the letter is devoted to the minimum uncertainty states (MUS) in search of squeezed and coherent states.

The Hamiltonian of the supersymmetric harmonic oscillator is given by [4]

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2x^2-\tfrac{1}{2}\omega\sigma_3\equiv H_{\mathrm{B}}-\tfrac{1}{2}\omega\sigma_3 \qquad (1)$$

where $H_{\mathrm{B}}|n\rangle=(n+\frac{1}{2})|n\rangle$ and σ_3 is the third component of the Pauli matrices. The eigenstates and the eigenvalues of H are given by

$$\begin{aligned}|\psi_n\rangle&=\alpha\left|\begin{smallmatrix}|n\rangle\\ \cdot\end{smallmatrix}\right\rangle+\beta\left|\begin{smallmatrix}\cdot\\ |n-1\rangle\end{smallmatrix}\right\rangle \qquad E_n=n\omega \qquad n=1,2,3,\ldots\\ |\psi_0\rangle&=\left|\begin{smallmatrix}|0\rangle\\ \cdot\end{smallmatrix}\right\rangle \qquad E_0=0.\end{aligned} \qquad (2)$$

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It was shown [3] that in this system one can define an annihilation operator:

$$A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \tag{3}$$

such that

$$A|\psi_n\rangle \propto |\psi_{n-1}\rangle \tag{4}$$

where a is the usual boson operator. The supercoherent states, defined as the eigenstates of A, i.e. $A|Z\rangle = z|Z\rangle$ are

$$|Z\rangle = a_0|z_f\rangle + c_1|\tilde{z}_s\rangle \tag{5}$$

where

$$|z_f\rangle \equiv \left|{}^{|z\rangle}_{\cdot}\right\rangle \qquad |\tilde{z}_s\rangle = \frac{1}{\sqrt{2}}\left|\begin{matrix} -(\partial/\partial z)|z\rangle \\ |z\rangle \end{matrix}\right\rangle \tag{6}$$

where a_0 and c_1 are constants, $|z\rangle$ being the standard boson coherent states. Another way of writing $|Z\rangle$ is as

$$|Z\rangle = \alpha|z_f\rangle + \beta|z_s\rangle \tag{7a}$$

with

$$|z_s\rangle = \frac{\bar{z}}{\sqrt{2}}|z_f\rangle + |\tilde{z}_s\rangle = \frac{1}{\sqrt{2}}\left|\begin{matrix} |\bar{z}\rangle - (\partial/\partial z)|z\rangle \\ |z\rangle \end{matrix}\right\rangle. \tag{7b}$$

where the bar denotes complex conjugation. The advantage of the form given by equation (7b) is that $\langle z_f|z_s\rangle = 0$.

As we know, the usual boson coherent states can be obtained as a displaced vacuum, with a unitary displacement operator. Our first question is: is there a unitary displacement operator for the supercoherent states? To answer this question we notice that from equations (5) and (6) one can write the supercoherent states as

$$|Z\rangle = [D(z)\otimes I]K(z)\left|{}^{|0\rangle}_{\cdot}\right\rangle \tag{8}$$

where $D(z) = \exp[za^+ - \bar{z}a]$ is the usual Glauber displacement operator and

$$K(z) = \begin{bmatrix} (a_0 - c_1\bar{z}/\sqrt{2}) - c_1 a^+/\sqrt{2} & P \\ c_1/\sqrt{2} & Q \end{bmatrix} \tag{9}$$

P and Q being arbitrary boson operators. If we want a unitary operator, that means $K^+(z)K(z) = 1$, which in turn implies

$$\begin{aligned} &(\beta - \lambda a^+)(\bar{\beta} + \bar{\lambda} a)PP^+ = 1 \\ &|\lambda|^2 + QQ^+ = 1 \\ &-(\beta + \lambda a^+)\bar{\lambda} + PQ^+ = 0 \end{aligned} \tag{10}$$

where $\beta \equiv a_0 + \lambda\bar{z}$, and $\lambda \equiv -c_1/\sqrt{2}$.

The set of equations (10) are inconsistent unless $\lambda = 0$. If $\lambda = 0$, $K(z) = \left({}^{1}_{\cdot}\ {}_{1}\right)$ up to an irrelevant phase factor. Therefore, the only unitary displacement operator is $D(z)\otimes I$, that generates a subset of the supercoherent states, namely the fermionic sector.

Next, we want to study the squeezing of the supercoherent states. For this purpose, we extend the usual definition of creation and annihilation operators:

$$a \to a \otimes I \qquad a^+ \to a^+ \otimes I \tag{11}$$

and therefore we have a similar extension for $\hat{x} = (1/\sqrt{2})(a + a^+)$ and $\hat{p} = (\mathrm{i}/\sqrt{2})(a^+ - a)$.

Aragone and Zypman [3] have shown that the fermionic sector ($|z_f\rangle$) are MUS with respect to $\hat{x}$ and $\hat{p}$. Also, if we consider the states orthogonal to the fermionic sector ($|z_s\rangle$), these are not MUS with respect to the two quadratures $\hat{x}$ and $\hat{p}$.

A rather tedious, but straightforward, calculation shows that

$$\begin{aligned} \langle(\Delta\bar{x})^2\rangle_s &= \tfrac{1}{2}(1+\cosh^2 r) + \tfrac{3}{2}\sinh^2 r + \sinh(2r)\cos\theta \\ \langle(\Delta\bar{p})^2\rangle_s &= \tfrac{1}{2}(1+\cosh^2 r) + \tfrac{3}{2}\sinh^2 r - \sinh(2r)\cos\theta \end{aligned} \tag{12}$$

where

$$\bar{x} = S(\alpha)\hat{x}S^+(\alpha) \qquad \bar{p} = S(\alpha)\hat{p}S^+(\alpha) \tag{13}$$

$S(\alpha)$ is the usual squeezing operator defined as [5]:

$$S(\alpha) = \exp[\tfrac{1}{2}(\alpha a^{+2} - \bar{\alpha}a^2)] \qquad \alpha = r\,\mathrm{e}^{\mathrm{i}\theta} \tag{14}$$

and the subscript s in equation (12) indicates that the averages have been taken with respect to the $|z_s\rangle$ states.

From equation (14), it is easy to prove that

$$S(\alpha)aS^+(\alpha) = a\cosh r + a^+\mathrm{e}^{\mathrm{i}\theta}\sinh r. \tag{15}$$

To derive the results given in equation (12), we made use of the above equality (15).

For $r = 0$ (no squeezing), we regain the results of [3]. For large r, we get a decrease of $\langle(\Delta\bar{p})^2\rangle_s$, but the lowest it can get is $\frac{1}{2}$. From these results we immediately see that the $S^+(\alpha)|z_s\rangle$ states are neither squeezed nor MUS, namely $\langle(\Delta\bar{x})\rangle_s\langle(\Delta\bar{p})\rangle_s$ is always larger than $\frac{1}{2}$.

At this point we would like to focus on a more detailed discussion on the MUS. A MUS of two Hermitian operators $\hat{A}$ and $\hat{B}$ is given by [6]

$$(\hat{A} + \mathrm{i}\lambda\hat{B})|\psi\rangle = \beta|\psi\rangle \tag{16}$$

where λ is real and positive and $\beta = \langle\hat{A}\rangle + \mathrm{i}\lambda\langle\hat{B}\rangle$. From equation (16), it is simple to prove that

$$(\Delta\hat{A})^2 = \frac{|\lambda|}{2}|\langle[\hat{A}, \hat{B}]\rangle| \qquad (\Delta\hat{B})^2 = \frac{1}{2|\lambda|}|\langle[\hat{A}, \hat{B}]\rangle|. \tag{17}$$

The states $|\psi\rangle$ defined by equation (16) are the minimum uncertainty states for $\hat{A}$ and $\hat{B}$. A subset of these states, those with $\lambda = \pm 1$ correspond to the coherent states, for which $(\Delta\hat{A})^2 = (\Delta\hat{B})^2$. If $\lambda \neq 1$, we generate squeezed MUS, where the fluctuation of one operator is reduced with respect to the other one, but their product

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 = \tfrac{1}{4}|\langle[\hat{A}, \hat{B}]\rangle|^2 \tag{18}$$

still corresponds to minimum uncertainty states. Now, if we specialise to $\hat{x}$ and $\hat{p}$ as defined at the beginning of this letter, we can write

$$(\hat{x} + \mathrm{i}\lambda\hat{p})|\psi\rangle = \beta|\psi\rangle \tag{19}$$

or

$$\left(\frac{1+\lambda}{2\sqrt{\lambda}}\,a-\frac{\lambda-1}{2\sqrt{\lambda}}\,a^+\right)|\psi\rangle=\frac{\beta}{\sqrt{2\lambda}}\,|\psi\rangle \tag{20}$$

and if we define

$$\cosh r\equiv\frac{1+\lambda}{2\sqrt{\lambda}}\qquad \sinh r=\frac{\lambda-1}{2\sqrt{\lambda}} \tag{21}$$

one can write

$$(a\cosh r-a^+\sinh r)|\psi\rangle=\frac{\beta}{\sqrt{2\lambda}}\,|\psi\rangle. \tag{22}$$

From equations (15) and (22), we can immediately see that ($\theta=\pi$)

$$S(-\alpha)aS^+(-\alpha)|\psi\rangle=\frac{\beta}{\sqrt{2\lambda}}\,|\psi\rangle \tag{23}$$

but since $S^+(\alpha)=S(-\alpha)$, then

$$aS(\alpha)|\psi\rangle=\frac{\beta}{\sqrt{2\lambda}}\,S(\alpha)|\psi\rangle \tag{24}$$

and we can write

$$S(\alpha)|\psi\rangle=|z\rangle_{\mathrm{N}}\qquad z=\frac{\beta}{\sqrt{2\lambda}} \tag{25}$$

where the subscript N refers to the fact that the coherent state is now normalised. From equation (24), we can immediately see that the squeezed minimum uncertainty states are generated by a unitary squeezing operator $S^+(\alpha)$ acting on a coherent state.

These well known results are quoted here because now we want to look at the minimum uncertainty states for supersymmetric operators. We would like to define a couple of operators X and P with properties different from the diagonal extension mentioned before. If we define them diagonal, as well as the squeezing operator, we get the result given by equation (12), that is, there is a reduction in the fluctuation of one of the quadratures, but not below the quantum noise level ($\frac{1}{2}$) and we do not have any squeezing. On the other hand, if one defines $|\psi\rangle=\left|\begin{smallmatrix}|\psi_1\rangle\\|\psi_2\rangle\end{smallmatrix}\right\rangle$ and proceeds as outlined in equations (19)-(24) with a and a^+ diagonal (i.e. $a\to\left(\begin{smallmatrix}a & \cdot\\ \cdot & a\end{smallmatrix}\right)$, $a^+\to\left(\begin{smallmatrix}a^+ & \cdot\\ \cdot & a^+\end{smallmatrix}\right)$), we can get the standard squeezed state for both $|\psi_1\rangle$ and $|\psi_2\rangle$. Now, we have defined A, which is a non-diagonal operator that acts as the annihilation operator for the supersymmetric harmonic oscillator. We then could define two new operators $\hat{X}$ and $\hat{P}$ as

$$\hat{X}=\frac{1}{\sqrt{2}}(\hat{A}^++\hat{A})\qquad \hat{P}=\frac{\mathrm{i}}{\sqrt{2}}(\hat{A}^+-\hat{A}) \tag{26}$$

which would be the 'natural' way of defining $\hat{X}$ and $\hat{P}$ in terms of the annihilation operator $\hat{A}$.

The next question is: can we find minimum uncertainty states for the operators $\hat{X}$ and $\hat{P}$ which can be either 'coherent' or squeezed in the sense of $\lambda=1$ or $\lambda\neq 1$ respectively? Notice that coherence here has a different meaning to the Aragone and Zipman [3] supercoherent state, since they obtain states that are not MUS (for $\hat{x}$ and

$\hat{p}$) and are eigenstates of the annihilation operator A. The basic equations for our MUS are

$$(\hat{X}+\mathrm{i}\lambda\hat{P})|\psi\rangle=\gamma|\psi\rangle$$
$$\hat{X}+\mathrm{i}\lambda\hat{P}\equiv\frac{1}{\sqrt{2}}\begin{pmatrix} a^+(1-\lambda)+a(1+\lambda) & 1+\lambda \\ 1-\lambda & a^+(1-\lambda)+a(1+\lambda)\end{pmatrix}$$
$$|\psi\rangle=\left|\begin{smallmatrix}|\psi_1\rangle\\|\psi_2\rangle\end{smallmatrix}\right\rangle. \tag{27}$$

To solve equations (27) for $|\psi\rangle$, we define basis states which are the eigenstates of the supersymmetric harmonic oscillator, namely:

$$|\psi_n\rangle=\left|\begin{smallmatrix}\alpha|n\rangle\\ \beta|n-1\rangle\end{smallmatrix}\right\rangle. \tag{28}$$

If we multiply equation (27) by $\langle\psi_n|$, and define

$$c_n\equiv\langle n|\psi_1\rangle \qquad d_n=\langle n|\psi_2\rangle \tag{29}$$

we get the following recursion relations:

$$c_{n+1}\bar{\alpha}(1+\lambda)\sqrt{n+1}+c_{n-1}(1-\lambda)(\bar{\alpha}\sqrt{n}+\bar{\beta})=\sqrt{2}\bar{\alpha}\gamma c_n$$
$$d_{n+1}(1+\lambda)(\bar{\alpha}+\bar{\beta}\sqrt{n+1})+d_{n-1}\bar{\beta}(1-\lambda)\sqrt{n}=\sqrt{2}\bar{\beta}\gamma d_n. \tag{30}$$

Although the general solution of the system (30) is complex, we discuss some simple cases.

(i) $\lambda=1, \beta=0$, gives trivially

$$|\psi\rangle_i=\left|\begin{smallmatrix}|z=\gamma/\sqrt{2}\rangle\\ \cdot\end{smallmatrix}\right\rangle$$

with an ordinary coherent state in the upper component of $|\psi\rangle$.

(ii) $\lambda=1, \alpha=0$, gives

$$|\psi\rangle_{ii}=\left|\begin{smallmatrix}\\ |z=\gamma/\sqrt{2}\rangle\end{smallmatrix}\right\rangle$$

with an ordinary coherent state in the lower component of $|\psi\rangle$.

(iii) In a similar way, $\lambda\neq 1$ and $\beta=0$ or $\alpha=0$, will give an ordinary squeezed state in the upper or lower component of $|\psi\rangle$ respectively.

(iv) A rather interesting case arises when $\alpha,\beta\neq 0$. Take $\lambda=1, \alpha=\beta\neq 0$. In a straightforward calculation we get

$$|\psi_1\rangle=|z=\gamma/\sqrt{2}\rangle \qquad \text{coherent state.}$$
$$|\psi_2\rangle=C_0\sum_{n=0}^{\infty}\left(\frac{\gamma}{\sqrt{2}}\right)^n\left(\prod_{p=1}^{n}(1+\sqrt{p})\right)^{-1}|n\rangle$$

with C_0 being a normalisation factor given by

$$C_0=\left[\sum_{n=0}^{\infty}\left(\frac{\gamma}{\sqrt{2}}\right)^{2n}\left(\prod_{p=1}^{n}(1+\sqrt{p})\right)^{-2}\right]^{-1/2}.$$

This would be a coherent state in the sense of equal fluctuations for $\hat{X}$ and $\hat{P}$. We have not been able to solve the case $\lambda\neq 1$ and $\alpha=\beta\neq 0$ since the standard technique of converting the difference equation into a differential one, via a generating function, does not work here, although one can always express c_n and d_n in terms of complicated continued fractions†.

† For a general discussion on the three-term recursion relations, see [7].

It is interesting to note that, as opposed to the ordinary harmonic oscillator, in the supersymmetric case, if one forms ωA^+A, it does not generate the Hamiltonian given in equation (1) but rather

$$H' = H_B - \tfrac{1}{2}\omega\sigma_3 + \omega(a^+\sigma_+ + a\sigma_-). \tag{31}$$

If one performs the transformation $\sigma_3 \rightarrow -\sigma_3$, $\sigma_\pm \rightarrow \sigma_\mp$, in this Hamiltonian, all the algebraic properties remain unchanged and H' becames the well known model of Jaynes and Cummings [8], for extremely strong coupling, $g = \omega$. This model has been widely used in quantum optics to describe the interaction of a two-level atom with one mode of the electromagnetic field. It is interesting to note that, although in the past this model was considered highly academic, it has been recently verified experimentally through micromaser experiments [9].

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